TOTALLY CLASSICAL CALOGERO MODEL

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Abstract

We show that the standard Calogero Lax matrix can be interpreted as a function on the fuzzy sphere and the Avan-Talon r-matrix as a function on the direct product of two fuzzy spheres. We calculate the limiting Lax function and r-function when the fuzzy sphere tends to the ordinary sphere and we show that they define an integrable model interpreted as a large N Calogero model by Bordemann, Hoppe and Theisen.

1 Introduction

The formalism based on the concepts of the Lax matrix and the Lax pair turns out to be very efficient in a study of a large class of classical completely integrable models [4, 7, 8, 33]. Briefly speaking, the Lax pair is a pair of matrix valued functions L, M on the phase space P of the integrable model such that the equations of motion of the latter can be written in terms of the matrix commutator

$$\frac{d}{dt}L = [L, M]. \tag{1.1}$$

Moreover, the Lax matrix L is required to generate integrals of motion in Poisson involution, i.e. it should fulfil a condition

$$\{\mathrm{tr}\, L^m, \mathrm{tr}\, L^n\}_P = 0, \quad \forall m, n. \tag{1.2}$$

Bordemann, Hoppe and Theisen [10] had an idea to generalize this formalism by replacing the matrices L, M by functions on some two-dimensional auxiliary compact symplectic manifold B and the commutator by the auxiliary Poisson bracket $\{., .\}_B$:

$$\frac{d}{dt}L = \{L, M\}_B. \tag{1.3}$$

Moreover, they imposed the following condition on L

$$\left\{\int_B \omega L^m, \int_B \omega L^n\right\}_P = 0, \quad \forall m, n, \tag{1.4}$$

where ω denotes the auxiliary symplectic form on B.

Starting with a suitable ansatz for the Lax function, Bordemann, Hoppe and Theisen [10] have identified several field theories in 1+1 dimensions which are integrable via their novel mechanism. Among them they singled out two models the classical actions of which read respectively

$$S_t = \frac{1}{2}\int dt \int_{-1}^{1} d\sigma \Big(\dot{q}(\sigma)^2 - a^2 \exp q'(\sigma)\Big), \tag{1.5}$$

$$S_c = \frac{1}{2}\int dt \int_{-1}^{1} d\sigma \left(\dot{q}(\sigma)^2 - \frac{a^2}{q'(\sigma)^2}\right). \tag{1.6}$$

Here the dependence of the dynamical field $q(\sigma)$ on the time t is tacitly understood, a is a coupling constant, the "dot" and "prime" denote the time and the space derivatives, respectively, and boundary conditions on $q(\sigma)$ will be specified in Section 3.

It is instructive to compare the field theories (1.5),(1.6) with the integrable N-particle Toda and Calogero systems:

$$S_{tN} = \frac{1}{2}\int dt \Big(\sum_{1\le i\le N} \dot{q}_i^2 - \kappa_N^2 \sum_{1\le i\le N} \exp(q_i - q_{i+1})\Big), \tag{1.7}$$

$$S_{cN} = \frac{1}{2}\int dt \Big(\sum_{1\le i\le N} \dot{q}_i^2 - \sum_{1\le i\ne j\le N} \frac{\kappa_N^2}{(q_i - q_j)^2}\Big), \tag{1.8}$$

The similarity between field theoretical and N-particle expressions was not the only reason why Bordemann, Hoppe and Theisen interpreted the models (1.5),(1.6) as "infinite dimensional analogues of N-particle Toda- and Calogero systems". In fact, although they did not derive these field theories as large N limits of the N-particle models, they argued that it should be possible to do it because the auxiliary Poisson algebra of functions on the two dimensional compact symplectic manifold B can be viewed as infinite-dimensional version $gl(\infty)$ of the algebra $gl(N)$ of finite size $N \times N$-matrices [27]. The parameter $N/2$ should be then interpreted as the Planck constant of an auxiliary quantization converting the auxiliary Poisson algebra into the matrix algebra and the integrable field theories (1.5),(1.6) could be therefore called "doubly classical" or "totally classical".

The suggestion of [10] to obtain the Lax functions of the totally classical field models (1.5),(1.6) as a large N limit of the known Lax matrices of the N-particle models (1.7),(1.8) was then realized by Hoppe in the Toda case [29]. However, to our best knowledge, a similar result was not obtained for the Calogero model. We wish to stress that the similarity between the field theoretical model and the finite N-particle theory is more evident for the Toda model than for the Calogero one. Indeed, the Toda field theory (1.5) is local in the sense that its Lagrangian contains only the first derivative $q'(\sigma)$ and the discrete theory (1.7) is also local in the sense that the potential contains only the interaction of the nearest neighbours. In reality, the N-particle Toda model (1.7) is nothing but a straightforward discretization of the Toda field theory (1.5).

In the Calogero case the situation is much more subtle, because the field theory (1.6) is local but the N-particle Calogero model (1.8) is non-local. Indeed, the Calogero model (1.8) is not a discretization of the field theory (1.6). On the top of it, there exists a *non-local* field theory referred to as the large N Calogero model in the literature [1, 41] the discretization of which does give the N-particle Calogero model (1.8). Would the existence of two different large N limits of the same finite N theory indicate a contradiction? A priori no, since much depends in which context the large N limit is taken, what other parameters get fixed while taking the limit etc. [5, 14, 21, 23, 31, 32, 36, 37]. We shall actually justify the interpretation of the model (1.6) as one particular large N limit by appropriately tuning the coupling constant κ_N and the Darboux symplectic Ω_N form of the N-particle Calogero model (1.8). Speaking more precisely, we shall prove that the discrete non-locality gets eliminated and the finite N Calogero model approaches the totally classical field theory (1.6) if we make both κ_N and Ω_N proportional to $1/N$. This will constitute the first result of the present paper.

In the original article [10], the fundamental involutivity relation (1.4) was obtained by a direct computation starting from an appropriate ansatz for the Lax function. In the finite N case, the involutivity (1.2) can be proved most easily by using the concept of the classical r-matrix. We recall that the classical r-matrix [7, 8] is a map $r(N)_{12}$ from the phase space of the model into the direct product of two copies of the matrix algebra such that a fundamental compatibility relation with the Lax matrix $L(N)$ holds:

$$\{L(N)_1, L(N)_2\}_P = -\frac{\mathrm{i}N}{2}[r(N)_{12}, L_1(N)] + \frac{\mathrm{i}N}{2}[r(N)_{21}, L_2(N)], \tag{1.9}$$

where $L(N)_1 = L(N) \otimes 1$ and $L(N)_2 = 1 \otimes L(N)$. It is then very easy to show that (1.9) implies (1.2).

In the totally classical context, Hoppe [28] has suggested to introduce a concept of an r-function on $P \times B \times B$, which would verify an analogue of (1.9):

$$\{L(z), L(w)\}_P = \{r(z,w), L(z)\}_B - \{r(w,z), L(w)\}_B. \tag{1.10}$$

Here L is the Lax function, the dependence of L and r on the coordinates of P is tacitly understood in (1.10) and the letter z or w stands for some parametrization of the auxiliary symplectic manifold B. As in the finite N case it is easy to show that the involutivity relation (1.4) is the consequence of (1.10).

In the case of the totally classical Calogero model, Hoppe started to look for the r-function by choosing an appropriate ansatz which he then substituted into the condition (1.10). Remarkably, the resulting object which he has found is not quite an r-function but rather an r-distribution in the variables parametrizing $B \times B$! This may look surprising at a first sight: how the large N-limit may convert matrices into distributions? As we shall see, this is indeed the case and the r-distribution does arise in the large N limit. The derivation of Hoppe's r-distribution directly from the Avan-Talon r-matrix [6] of the N-particle Calogero model constitutes the second result of the present article.

In this paper we choose the ordinary sphere as the auxiliary symplectic manifold B and in Section 2 we review its quantization called the fuzzy sphere [27, 35]. Then in Sections 3 and 4 we prove that the Lax matrix and r-matrix of the N-particle Calogero model are respectively the fuzzy quantizations of the Lax function [10] and of the r-distribution [28] of the totally classical Calogero model (1.6). We finish by a short outlook.

2 The fuzzy sphere

The ordinary symplectic sphere S^2 is a surface embedded in $\mathbb{R}^3$ defined in Carthesian coordinates x_1, x_2, x_3 as

$$x_1^2 + x_2^2 + x_3^2 = 1. \tag{2.11}$$

We parametrize it as

$$x_1 = \sqrt{1-\sigma^2}\cos\phi, \quad x_2 = \sqrt{1-\sigma^2}\sin\phi, \quad x_3 = \sigma, \qquad \sigma \in [-1,1], \phi \in [-\pi,\pi]. \tag{2.12}$$

The standard round symplectic form on S^2 is then given by

$$\omega \equiv -\frac{1}{2}\epsilon_{jkl}x_j dx_k \wedge dx_l \Big|_{S^2} = d\sigma \wedge d\phi. \tag{2.13}$$

The result of the quantization of the symplectic manifold (S^2, ω) is known as "the fuzzy sphere" [27, 35]. The linear $SO(3)$-equivariant quantization map Q_N associates to smooth functions f on S^2 sequences of $N \times N$-matrices $Q_N(f)$ which are called the quantized or fuzzy functions. We shall not need an explicit formula for the quantization map Q_N but we do need three basic properties of it:

$$Q_N(f)Q_N(g) = Q_N\left(fg + O\left(\frac{2}{N}\right)\right), \tag{2.14}$$

$$[Q_N(f), Q_N(g)] = Q_N\left(\mathrm{i}\frac{2}{N}\{f,g\}_B + O\left(\frac{4}{N^2}\right)\right), \tag{2.15}$$

$$\frac{1}{2\pi}\int_{S^2}\omega f = \mathrm{tr}\, Q_N\left(\frac{2}{N}f + O\left(\frac{4}{N^2}\right)\right). \tag{2.16}$$

Obviously the parameter $2/N$ plays the role of the auxiliary Planck constant.

To give a flavor, what the map Q_N is about, let us make explicit the quantized versions of the functions x_3, $x_1 \pm \mathrm{i}x_2$ defined in (2.12):

$$Q_N(x_3)_{ij} = \frac{1}{\sqrt{N^2-1}}(N+1-2j)\delta_{ij}, \quad Q_N(x_1+\mathrm{i}x_2)_{ij} = \frac{2}{\sqrt{N^2-1}}\sqrt{(j-1)(N-j+1)}\delta_{i,j-1} \tag{2.17}$$

and $Q_N(x_1 - \mathrm{i}x_2)$ is the Hermitian-conjugated matrix $Q_N(x_1+\mathrm{i}x_2)^\dagger$. In particular, it is then easy to verify that it holds the emblematic fuzzy sphere relation

$$Q_N(x_1)^2 + Q_N(x_2)^2 + Q_N(x_3)^2 = \mathbf{1}_N, \tag{2.18}$$

where $\mathbf{1}_N$ stands for the unit $N \times N$-matrix.

So far we have learned that every smooth function on the sphere gives rise to a sequence of $N \times N$-matrices. It is perhaps less known that appropriate sequences of $N \times N$-matrices may represent quantizations of not just smooth functions, but also of singular functions and/or distributions on S^2. As an example particularly relevant for the present paper, we now describe the fuzzy vortices. For that, consider the sequence of $N \times N$ matrices $V(N)$ defined by their matrix elements:

$$V(N)_{ij} = \delta_{i,j-1}, \quad i,j = 1,...,N. \tag{2.19}$$

In words: $V(N)$ is the Jordan block with zeros on the principal diagonal. We immediately check that

$$Q_N(x_1+\mathrm{i}x_2) = \sqrt{1-\frac{\sqrt{N-1}}{\sqrt{N+1}}Q_N(x_3)}\sqrt{1+\frac{\sqrt{N+1}}{\sqrt{N-1}}Q_N(x_3)}V(N) \tag{2.20}$$

and we note that the diagonal matrices of which we take the square roots have all eigenvalues strictly positive. In the limit $N \to \infty$, we then obtain

$$x_1 + \mathrm{i}x_2 = \sqrt{1-x_3^2}\; V_\infty, \tag{2.21}$$

which together with (2.12) leads to an identification of V_∞ with the ordinary sphere vortex configuration $e^{\mathrm{i}\phi}$ and to an extension of the quantization map to this vortex configuration by setting

$$Q_N(e^{\mathrm{i}\phi}) = V(N). \tag{2.22}$$

Another relevant quantized singular object on the sphere is characterized by the following sequence of $N \times N$ matrices $K(N)$:

$$K(N) := \sum_{k,l}^{N} E_{kl}, \tag{2.23}$$

where E_{kl} is the elementary matrix with 1 in k^{th} row and l^{th} column and 0 everywhere else. Note that $K(N)$ are matrices with all elements equal to 1. For a finite N, we observe

$$K(N) = (V(N)^\dagger)^{N-1} + ... + (V(N)^\dagger)^2 + V(N)^\dagger + \mathbf{1}_N + V(N) + V(N)^2 + ... + V(N)^{N-1} \tag{2.24}$$

which gives from (2.22)

$$K_\infty = \lim_{N\to\infty} Q_N\left(\sum_{j=1-N}^{N-1} e^{\mathrm{i}j\phi}\right) = 2\pi\delta(\phi). \tag{2.25}$$

Said differently, we extend the quantization map to the delta function by setting

$$2\pi Q_N(\delta(\phi)) = K(N). \tag{2.26}$$

Now we use the same symbol Q_N for the quantization of the direct product $S^2 \times S^2$. We can argue that

$$Q_N\left(\delta(\sigma_1-\sigma_2)\delta(\phi_1-\phi_2)\right) = \frac{N}{4\pi}\sum_{k,l}^{N} E_{kl}\otimes E_{lk}. \tag{2.27}$$

Indeed, the δ-function $\delta(\sigma_1-\sigma_2)\delta(\phi_1-\phi_2)$ on $S^2\times S^2$ is characterized by the property

$$\int_{-1}^{1} d\sigma_2 \int_{-\pi}^{\pi} d\phi_2 \delta(\sigma_1-\sigma_2)\delta(\phi_1-\phi_2) f(\sigma_2,\phi_2) = \int \omega_2 \delta(\sigma_1-\sigma_2)\delta(\phi_1-\phi_2) f(\sigma_2,\phi_2) = f(\sigma_1,\phi_1), \tag{2.28}$$

where $f(\sigma_2,\phi_2)$ is an arbitrary smooth function on S^2. Following (2.16), the quantized version of (2.28) is

$$\frac{4\pi}{N}\mathrm{tr}_2\left(Q_N(\delta(\sigma_1-\sigma_2)\delta(\phi_1-\phi_2))(\mathbf{1}_N\otimes Q_N(f))\right) = Q_N(f). \tag{2.29}$$

On the other hand, for the $N\times N$-matrix $Q_N(f)$ it obviously holds

$$\mathrm{tr}_2\left((\sum_{k,l}^{N} E_{kl}\otimes E_{lk})(\mathbf{1}_N\otimes Q_N(f))\right) = Q_N(f). \tag{2.30}$$

Comparing the last two equalities leads to the identification (2.27).

Finally, we shall need also the fuzzy version of the "diagonal" delta function $\delta(\sigma_1-\sigma_2)$ on $S^2\times S^2$. Since the fuzzy version of a ϕ-independent function on S^2 is a diagonal matrix, the fuzzy version of $\delta(\sigma_1-\sigma_2)$ must be given as a sum of direct products of diagonal matrices. To find out which diagonal matrices appear in those direct products we look for a fuzzy analogue of the following identity:

$$\frac{1}{2\pi}\int_{-1}^{1} d\sigma_2 \int_{-\pi}^{\pi} d\phi_2 \delta(\sigma_1-\sigma_2) f(\sigma_2) = \frac{1}{2\pi}\int \omega_2 \delta(\sigma_1-\sigma_2) f(\sigma_2) = f(\sigma_1). \tag{2.31}$$

It obviously reads

$$\frac{2}{N}\mathrm{tr}_2\left(Q_N(\delta(\sigma_1-\sigma_2))(\mathbf{1}_N\otimes Q_N(f))\right) = Q_N(f). \tag{2.32}$$

We have also the following matrix identity:

$$\mathrm{tr}_2\left((\sum_{k}^{N} E_{kk}\otimes E_{kk})(\mathbf{1}_N\otimes Q_N(f))\right) = Q_N(f), \tag{2.33}$$

which holds for the diagonal matrix $Q_N(f)$. Comparing the last two equalities leads to the desired quantization formula

$$Q_N(\delta(\sigma_1 - \sigma_2)) = \frac{N}{2} \sum_k^N E_{kk} \otimes E_{kk}. \tag{2.34}$$

3 Large N limit of the Calogero Lax matrix

Recall first the standard formula for the Lax matrix $L(N)$ of the Calogero model [38]

$$L(N)_{ij} = p_i \delta_{ij} + (1 - \delta_{ij}) \frac{i\kappa_N}{q_i - q_j}, \quad 1 \leq i, j, \leq N, \tag{3.35}$$

where p_j, q_j are the Darboux coordinates on the phase space of the model. In what follows, we normalize the Darboux symplectic structure Ω_N and the coupling constant κ_N as

$$\Omega_N = \frac{2}{N} dq_j \wedge dp_j, \quad \kappa_N = \frac{c}{N}. \tag{3.36}$$

We now introduce important diagonal $N \times N$-matrices $R(N)$ and $P(N)$ as follows

$$R(N)_{ij} := q_i \delta_{ij}, \quad P(N)_{ij} := p_i \delta_{ij}, \quad i, j = 1, ..., N \tag{3.37}$$

and we naturally interpret $R(N)$ as $Q_N(q(\sigma))$ and $P(N)$ as $Q_N(p(\sigma))$. Speaking more precisely, we have from (2.17)

$$q_j \equiv Q_N(q(\sigma))_{jj} = q\left(\frac{N + 1 - 2j}{\sqrt{N^2 - 1}}\right) \tag{3.38}$$

and similarly for p_j.

Few comments are in order to justify the interpretation of $R(N)$ as $Q_N(q(\sigma))$ and $P(N)$ as $Q_N(p(\sigma))$. First of all, $q(\sigma), p(\sigma)$ are viewed as the functional coordinates of the *field theoretical* phase space of the totally classical Calogero model (1.6) with the Darboux Poisson bracket

$$\{p(\sigma_1), q(\sigma_2)\}_P = \delta(\sigma_1 - \sigma_2), \tag{3.39}$$

but at the same time they are viewed as ϕ-independent functions on the sphere. We note also that for j running through the set $1, ..., N$ the argument $\frac{N+1-2j}{\sqrt{N^2-1}}$ in (3.38) runs equidistantly through the interval $[-1, 1]$ which is indeed the domain of definition of the function $q(\sigma)$. Moreover, for neighbouring j and $j + 1$ the distance of the arguments is $\frac{2}{\sqrt{N^2-1}}$ which for large N is nothing but the B-Planck constant. We thus observe that the phase space P of the totally classical Calogero model (1.6) gets drastically shrunk by the fuzzification. Indeed, among all functions $q(\sigma), p(\sigma)$ which constitute points in P, only functions constant on the equidistant intervals of the lenghts $\frac{2}{\sqrt{N^2-1}}$ survive the fuzzification and form a $2N$-dimensional phase space $P(N)$ of the N-particle Calogero model. Moreover, we can derive from the Darboux Poisson bracket (3.39) on the phase space P the Poisson bracket on the $2N$-dimensional phase space $P(N)$ parametrized by $q_j, p_j,\ j = 1, ..., N$. To do that, we consider a function $T(\sigma)$ on the sphere and its diagonal quantization $Q_N(T)$. From the Darboux Poisson structure (3.39), we obtain

$$\left\{p(\sigma), \frac{1}{2\pi}\int \omega T q\right\}_P = T(\sigma). \tag{3.40}$$

Now from the property (2.16) we see that the fuzzification of the formula (3.40) must give

$$\left\{p_i, \frac{2}{N}\mathrm{tr}\,(Q_N(T)R(N))\right\}_{P(N)} = Q_N(T)_{ii}. \tag{3.41}$$

We thus infer

$$\{p_i, q_j\}_{P(N)} = \frac{N}{2}\delta_{ij} \tag{3.42}$$

which is consistent with the first formula in (3.36).

Let us verify that the N-particle Calogero Lax matrix $L(N)$ (3.35) has as a large N-limit some classical Lax observable L. Said differently, we shall show that $L(N)$ can be interpreted as the fuzzification $Q_N(L)$ of some function L on the ordinary sphere.

It is now easy to verify with the help of (2.26) that

$$[R(N), L(N)] = \mathrm{i}\kappa_N(K(N) - \mathbf{1}_N) = \frac{\mathrm{i}c}{N}Q_N(2\pi\delta(\phi) - 1), \tag{3.43}$$

where the matrix $K(N)$ was defined in (2.23). If it is true that $L(N)$ can be identified with $Q_N(L)$ for some L then we could rewrite (3.43) as

$$[Q_N(q(\sigma)), Q_N(L)] = \frac{\mathrm{i}c}{N}Q_N(2\pi\delta(\phi) - 1). \tag{3.44}$$

which, due to the fundamental quantization property (2.15), would lead to

$$\{q(\sigma), L(\sigma,\phi)\}_B = -q'(\sigma)\partial_\phi L = \frac{c}{2}(2\pi\delta(\phi) - 1). \tag{3.45}$$

And indeed! The differential condition (3.45) has the following obvious solution

$$L(\sigma,\phi) = p(\sigma) + \frac{c}{2\rho'(\sigma)}(\phi - \pi\mathrm{sign}(\phi)). \tag{3.46}$$

which coincides[1] for $c = 2\sqrt{3}a/\pi$ with the Lax function of the model (1.6) as found in [10]. Moreover, the condition (3.45) determines the function L almost unambiguously. The only ambiguity consists in adding a ϕ-independent function to the solution (3.46), however, this ambiguity is fixed by the fact that the diagonal term $p_i\delta_{ij}$ in the Lax matrix $L(N)$ must be equal to $Q_N(p(\sigma))$. Thus we have justified the interpretation [10] of the totally classical Calogero model as the large N-limit of the N-particle Calogero model.

Remark: It is important to stress that, as it stands, the formula (3.46) defines only a function on a subset of S^2 covered by the coordinate chart (σ,ϕ) and we need the classical Lax

[1] The comparison of (3.46) with the result of [10] must take into account the range of the parameter ϕ which is $[-\pi,\pi]$ in our paper and it was $[0,2\pi]$ in [10]. We have opted for the different range in order to stress that the Lax function L is discontinuous. This fact is indeed less visible in [10] since the discontinuity occurs precisely on the boundaries of the range $[0,2\pi]$.

observable everywhere on S^2. Fortunately, the function $\phi - \pi\text{sign}(\phi)$ smoothly extends to the anti-Greenwich meridian $\phi = \pm\pi$ and we can also extend $L_\infty(p,\rho;\sigma,\phi)$ to the poles $\sigma = \pm 1$ if we impose the following boundary conditions

$$\lim_{\sigma\to\pm 1}\rho'(\sigma) = +\infty. \tag{3.47}$$

It can be also checked that these somewhat exotic boundary conditions are consistent with the dynamics of the model (1.6) since they make sure that there is no flow of energy through the boundaries $\sigma = \pm 1$.

4 Large N limit of the Avan-Talon r-matrix

Consider an r-matrix $r(N)$ given by the formula

$$r(N)_{12} = \sum_{k\neq l}^{N} \frac{\mathrm{i}}{q_l - q_k} E_{kl} \otimes E_{lk} + \frac{1}{2}\sum_{k\neq l}^{N} \frac{\mathrm{i}}{q_l - q_k} E_{kk} \otimes (E_{kl} - E_{lk}). \tag{4.48}$$

Using the Poisson brackets (3.42), it is a matter of straightforward computation to verify that the r-matrix (4.48) and the Lax matrix $L(N)$ satisfy the following crucial property (cf.(1.9))

$$\{L(N)_1, L(N)_2\}_{P(N)} = -\frac{\mathrm{i}N}{2}[r(N)_{12}, L_1(N)] + \frac{\mathrm{i}N}{2}[r(N)_{21}, L_2(N)]. \tag{4.49}$$

We recall that it is this property (4.49) that guarantees the $P(N)$-Poisson commutativity of the traces $\mathrm{tr}\, L(N)^n$. For a computational convenience, the matrix $r(N)$ slightly differs from the r-matrix proposed by Avan and Talon in [6]; in fact, $r(N)$ is just another element of a moduli space [20] of all r-matrices verifying (4.49).

Recalling the matrix $R(N)$ introduced in the previous section it is easy to check that

$$[R(N)_1, r(N)_{12}] = -\mathrm{i}\sum_{k\neq l}^{N} E_{kl} \otimes E_{lk}; \tag{4.50}$$

$$[R(N)_2, r(N)_{12}] = \mathrm{i}\sum_{k,l}^{N} E_{kl} \otimes E_{lk} - \frac{\mathrm{i}}{2}\left[\left(\sum_{m}^{N} E_{mm}\otimes E_{mm}\right), \left(\mathbf{1}_N \otimes \sum_{k,l}^{N} E_{kl}\right)\right]_+, \tag{4.51}$$

where as usual $R(N)_1 = R(N)\otimes 1$, $R(N)_2 = 1\otimes R(N)$ and $[.,.]_+$ stands for an anticommutator. Now writing $R(N)$ as $Q_N(q(\sigma))$, using the formulae (2.26),(2.27) and (2.34) and supposing that $r(N)$ can be written as a fuzzification $Q_N(r)$ of some function r on $S^2\times S^2$, the equations (4.50) and (4.51) can be rewritten as

$$[Q_N(q(\sigma))_1, Q_N(r)] = -\frac{2\mathrm{i}}{N} Q_N\Big(2\pi\delta(\sigma_1-\sigma_2)\delta(\phi_1-\phi_2) - \delta(\sigma_1-\sigma_2)\Big), \tag{4.52}$$

$$[Q_N(q(\sigma))_2, Q_N(r)] = \frac{2\mathrm{i}}{N} Q_N\Big(2\pi\delta(\sigma_1-\sigma_2)\delta(\phi_1-\phi_2)\Big) - \frac{\mathrm{i}}{N}\Big[Q_N(\delta(\sigma_1-\sigma_2)), Q_N(2\pi\delta(\phi_2))\Big]_+. \tag{4.53}$$

Using the fundamental quantization properties (2.14) and (2.15), we deduce from (4.52) and (4.53) that r must fulfil

$$\{q(\sigma_1), r(\sigma_1,\phi_1;\sigma_2,\phi_2)\}_B = -q'(\sigma_1)\partial_{\phi_1} r = \delta(\sigma_1-\sigma_2)(1-2\pi\delta(\phi_1-\phi_2)). \tag{4.54}$$

$$\{q(\sigma_2), r(\sigma_1,\phi_1;\sigma_2,\phi_2)\}_B = -q'(\sigma_2)\partial_{\phi_2} r = 2\pi\delta(\sigma_1-\sigma_2)(\delta(\phi_1-\phi_2)-\delta(\phi_2)). \tag{4.55}$$

The relations (4.54) and (4.55) are indeed verified by the following distribution

$$r(\sigma_1,\phi_1;\sigma_2,\phi_2) = -\frac{1}{q'(\sigma_1)}\delta(\sigma_1-\sigma_2)\Bigg(E(\phi_1-\phi_2)+E(\phi_2)\Bigg). \tag{4.56}$$

Here E is viewed as a 2π-periodic function on the whole real axis $\mathbb{R}$ which is given by the expression $\phi - \pi\mathrm{sign}(\phi)$ when restricted to the interval $[-\pi,\pi]$:

$$E(\phi) := \phi - \pi\mathrm{sign}(\phi), \quad \phi\in[-\pi,\pi]. \tag{4.57}$$

The distribution (4.56) does coincide with the one found in [28], moreover, the first order differential conditions (4.54) and (4.55) determine r almost unambiguously. The only ambiguity consists in adding a ϕ_1,ϕ_2-independent function to the solution (4.56) of (4.54) and (4.55), however, such function would be a large N limit of a bi-diagonal term which is absent in $r(N)$. We thus conclude that the classical Yang-Baxter observable r (4.56) of the totally classical Calogero model (1.6) is indeed the large N-limit of the Avan-Talon r-matrix $r(N)_{12}$.

5 Outlook

As it is well-known, the Calogero model has attracted a lot of attention in pure mathematics and in mathematical physics (for reviews see [3, 18, 19, 39, 40, 43, 44]) since it appears in a large variety of contexts [2, 9, 11, 12, 13, 16, 17, 22, 24, 25, 26, 30, 34, 36, 42, 45, 46, 47, 48, 49, 50] ranging from condensed matter physics, higher spin algebras, two-dimensional QCD or fluid dynamics to microscopic description of black holes etc. We expect a relevance of the totally classical Calogero model in many of these contexts. Speaking more generally, it would be interesting to construct the totally classical models corresponding to various trigonometric, elliptic or even relativistic deformations of the Calogero one. This would certainly contribute to a better understanding of the duality properties of the integrable systems. One can also look for gauge transformations of the r-functions in order to get rid of the q dependence of r (4.56), much in the spirit of [20] which does that for the standard r-matrices.

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